

The Transverse Modes of a String with Variable Mass Density

T. M. Kalotas, A. R. Lee

Department of Physics, La Trobe University, Bundoora,

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Summary

The normal modes of transverse oscillation of a string of varying linear mass density are investigated by approximating the density function with segments of constant value. The computer implementation of the resulting 2×2 matrix formalism yielding the characteristic frequencies and associated standing-wave patterns, leads to the straightforward mode analysis of strings with arbitrary density profiles. The method also supplies a connection between the perturbations in characteristic frequency and the small changes in string density which cause them, thereby allowing mode frequencies to be manipulated in a controlled way through the density function. Computed examples are given to demonstrate all results.

Die Transversalmoden einer Saite mit variabler Massendichte

Zusammenfassung

Es wurden die Normalmoden der transversalen Schwingungen einer Saite veränderlicher Massendichte untersucht, indem die Dichtefunktion durch Abschnitte konstanter Dichte angenähert wurde. Die Durchführung eines 2×2 -Matrix-Formalismus mit dem Computer, wel-

che die Eigenfrequenzen und die zugehörigen Schwingungsverteilungen ergibt, führt direkt zur Modenanalyse von Saiten mit beliebigen Dichteprofilen. Die Methode liefert auch eine Verknüpfung zwischen den Störungen der Eigenfrequenzen aufgrund kleiner Änderungen der Saitendichte und ermöglicht damit die kontrollierte Manipulation der Eigenfrequenzen mittels der Dichtefunktion. Alle Ergebnisse werden durch Rechenbeispiele veranschaulicht.

Les modes transversaux d'une corde à densité de masse variable

Sommaire

Une étude des modes normaux de vibration transversale d'une corde à variation linéaire de densité de masse a été effectuée en simulant la fonction de densité par des segments de valeurs constantes. La résolution par ordinateur de la matrice 2×2 résultant de la mise en équation, et fournissant les fréquences propres et les modes de vibration propres associés, rend possible une analyse modale immédiate des cordes à profil de densité quelconque. La méthode fournit également la relation entre les perturbations fines de fréquences propres et les variations légères de densité qui en sont la cause, ce qui permet de manipuler les fréquences propres en jouant sur la fonction de densité. On fournit des exemples de calcul pour démontrer ces résultats.

1. Introduction

Musical instruments currently in use today which supply the desired frequencies to a sounding board or electric pick-up via the natural modes of oscillation of a clamped string under tension, invariably utilize strings with a uniform mass density. The qualitative features of the transverse vibrational motion of such strings have been well understood virtually from time immemorial, while their corresponding mathematical mode analysis turns out to be about as simple as it can be for any vibrating system. This simplicity, reflected in the natural emergence of a harmonic series of overtones is, however, crucially dependent on string uniformity. Once the linear mass density parameter

varies along the string, the analysis is no longer trivial. Notwithstanding the extra degree of complexity, it is nevertheless of interest to study such non-uniform string modes, both for the purpose of gauging the extent to which small deviations from uniformity can influence these modes, and also from the viewpoint of the design and construction of new string instruments.

With this in mind, we show here how to reduce the daunting differential equation analysis associated with a clamped non-uniform string, to a much simpler problem of pure algebra involving only 2×2 matrices. We achieve this by first approximating a non-uniform string by one made up of a sequence of N segments in each of which the density parameter takes a fixed value. Clearly with a large enough N one may in this way approach any given variable density as closely as desired. Then within each segment, the wave equation for the string displacement function is exactly solved apart from a pair of arbitrary constants. All that remains is to adjust these so that at any segment

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T. M. Kalotas, A. R. Lee, Department of Physics, La Trobe University, Bundoora, Victoria, Australia 3083.

boundary the displacement and its slope on one side matches the displacement and slope on the other. The resulting series of 2×2 matrix equations which achieve this matching is then further restricted by the overall boundary conditions that the ends of the string remain permanently at rest, and this in turn leads to the desired condition for the mode frequencies. Although the resulting condition has a considerably involved transcendental form, it is nevertheless readily processed by a computer program and yields beside the characteristic frequencies, also the corresponding string displacement functions in a straightforward way.

A further bonus of our algebraic approach is that a linear mathematical relationship can be established between small changes in the string density that one may wish to introduce into the various segments and the corresponding shifts in the mode-frequencies resulting from these changes. Such a relationship allows one to compute the changes in string density required to shift certain important mode frequencies of a stringed instrument by small predetermined amounts, a result which may lead to the creation of interesting new musical effects. In particular the selective loading of a uniform string may be studied with a view to manipulating its existing harmonic overtone structure.

In the following section we present the analytic-to-algebraic reduction as adapted from our similar analysis of one-dimensional quantum scattering [1], and we derive the algebraic conditions for normal modes leading to the corresponding standing wave patterns. In section 3 we illustrate the procedure by applying it to the simple cases of strings of one and two segments, and follow this in section 4 with a discussion of the computer implementation that is necessary when the number of segments N is large. Such a large N calculation for a sample analytic string density function is given in section 5. Following this we establish in section 6 the linear relationship between small changes in segment densities and the resulting shifts in mode-frequencies with an appropriate example to illustrate its use for design purposes.

2. Mathematical reductions of the boundary problem

We consider a string of length L lying under tension T with its ends at coordinates $x = X$ and $x = X + L$ and having in general a variable linear mass density $\mu(x)$. With the aid of the local speed of transverse wave propagation

$$c(x) \equiv \sqrt{\frac{T}{\mu(x)}} \quad (1)$$

the instantaneous string displacement $\chi(x, t)$ is specified by the wave equation

$$\left(\frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \chi(x, t) = 0. \quad (2)$$

On assuming a fixed angular frequency ω through writing

$$\chi(x, t) \equiv e^{i\omega t} \psi(x), \quad (3)$$

the wave eq. (2) reduces to the Helmholtz equation

$$\left[\frac{d^2}{dx^2} + k^2(x) \right] \psi(x) = 0, \quad (4)$$

with

$$k(x) \equiv \frac{\omega}{c(x)}. \quad (5)$$

At this point we break up the density function $\mu(x)$ into N constant segments $\mu_1, \mu_2, \dots, \mu_N$ as shown schematically in Fig. 1, with the segment widths $w_1, w_2, \dots, w_N$ (non-equal in general) satisfying

$$w_1 + w_2 + \dots + w_N = L, \quad (6)$$

and boundaries located at $X_1 (\equiv X), X_2, X_3, \dots, X_{N+1} (\equiv X + L)$ with

$$X_{j+1} = X_j + w_j, \quad j = 1, 2, \dots, N. \quad (7)$$

The fixed densities μ_j are determined by a linear interpolation rule

$$\mu_j = \frac{\mu(X_j) + \mu(X_{j+1})}{2}, \quad j \equiv 1, 2, \dots, N \quad (8)$$

or any other convenient averaging method. In this way we obtain via (1) and (5) the series of constant wavenumbers

$$k_j \equiv \omega \sqrt{\frac{\mu_j}{T}}, \quad j = 1, 2, \dots, N, \quad (9)$$

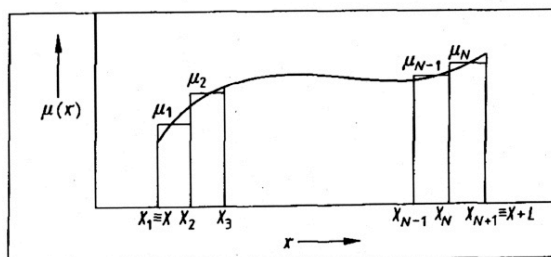


Fig. 1. Approximation of a continuous string density function $\mu(x)$ by segments of finite width.

with the aid of which eq. (4) reduces, in each segment j , to the simple harmonic form

$$\left[\frac{d^2}{dx^2} + k_j^2 \right] \psi_j(x) = 0, \quad L_j \leq x \leq L_{j+1}. \quad (10)$$

The displacement functions $\psi_j(x)$ follow immediately as

$$\psi_j(x) = a_j \cos k_j(x - X_j) + b_j \frac{\sin k_j(x - X_j)}{k_j} \quad (11)$$

with a_j, b_j arbitrary constants. Here by dividing the sine term of (11) by k_j , we guarantee that $\psi_j(x)$ continues to be a solution of (10) even when $k_j = 0$, in which case it reduces to the required form $\psi_j(x) = (\text{constant})x + (\text{constant})$.

For an acceptable complete solution, the individual segment function ψ_j of (11) must be joined so that both $\psi_j(x)$ and its derivative $\psi'_j(x)$ are continuous across segment boundaries. To achieve this conveniently we combine the expressions for $\psi_j(x)$ and $\psi'_j(x)$ into a 2-vector equation as

$$\begin{pmatrix} \psi_j(x) \\ \psi'_j(x) \end{pmatrix} = Z[k_j, -(x - X_j)] \begin{pmatrix} a_j \\ b_j \end{pmatrix}, \quad (12)$$

with the 2×2 matrix $Z[k, w]$ defined as

$$Z[k, w] \equiv \begin{pmatrix} \cos k w & -(\sin k w)/k \\ k \sin k w & \cos k w \end{pmatrix}. \quad (13)$$

Continuity then demands that at each of the boundaries $x = X_{j+1}$ $j = 1, 2, \dots, N-1$ we satisfy

$$\begin{pmatrix} \psi_j(X_{j+1}) \\ \psi'_j(X_{j+1}) \end{pmatrix} = \begin{pmatrix} \psi_{j+1}(X_{j+1}) \\ \psi'_{j+1}(X_{j+1}) \end{pmatrix}, \quad j = 1, 2, \dots, N-1, \quad (14)$$

which with the aid of (12) translates into the series of matrix equations for the a, b -coefficients as

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = Z[k_j, w_j] \begin{pmatrix} a_{j+1} \\ b_{j+1} \end{pmatrix}, \quad j = 1, 2, \dots, N-1. \quad (15)$$

From (15) we readily obtain through successive substitutions, the relation between a_1, b_1 and a_N, b_N as

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \tilde{\Pi} \begin{pmatrix} a_N \\ b_N \end{pmatrix}, \quad (16)$$

where

$$\tilde{\Pi} \equiv Z[k_1, w_1] Z[k_2, w_2] \dots Z[k_{N-1}, w_{N-1}]. \quad (17)$$

With continuity thus guaranteed, it only remains to impose the boundary condition that the string ends be permanently undisplaced. To achieve this we need

$$\psi_1(X_1) = a_1 = 0, \quad (18)$$

$$\psi_N(X_{N+1}) = a_N \cos k_N w_N + \frac{n_N \sin k_N w_N}{k_N} = 0. \quad (19)$$

Then on substituting (18) into (16) and solving for a_N, b_N (using the property that each Z -matrix and hence also $\tilde{\Pi}$ is unimodular), we obtain

$$\begin{pmatrix} a_N \\ b_N \end{pmatrix} = \tilde{\Pi}^{-1} \begin{pmatrix} 0 \\ b_1 \end{pmatrix} = \begin{pmatrix} \tilde{\Pi}_{22} & -\tilde{\Pi}_{12} \\ -\tilde{\Pi}_{22} & \tilde{\Pi}_{11} \end{pmatrix} \begin{pmatrix} 0 \\ b_1 \end{pmatrix} = b_1 \begin{pmatrix} -\tilde{\Pi}_{12} \\ \tilde{\Pi}_{11} \end{pmatrix}. \quad (20)$$

Since b_1 cannot be zero for a nontrivial solution, we have on combining (19) and (20) the desired mode condition

$$\tilde{\Pi}_{12} \cos k_N w_N - \tilde{\Pi}_{11} \frac{\sin k_N w_N}{k_N} = 0. \quad (21)$$

An alternative way to write this condition is to define the related matrix Π via

$$\Pi = \tilde{\Pi} \cdot Z[k_N, w_N] = Z[k_1, w_1] Z[k_2, w_2] \dots Z[k_N, w_N], \quad (22)$$

in terms of which (21) takes the simpler equivalent form

$$F(\omega) \equiv \Pi_{12} = 0. \quad (23)$$

The criterion (23) represents a transcendental equation for the mode frequencies ω as can be seen by multiplying out the Z -matrix product of (22). The complexity of the function F grows rapidly with N ; for the purposes of illustration we display it in the next section explicitly for the simplest cases $N = 1, 2$. To obtain accurate numerical values for the mode frequencies themselves, it is necessary for all $N \geq 2$ to resort to a computer strategy for solving (23). This will be described fully in section 4.

3. One and two segment strings

Here we illustrate the simplicity of application of the general mode condition (23) for $N = 1, 2$. For the trivial case $N = 1$, which is just that of the well-known uniform string of constant mass density μ_1 , we have

$$w_1 = L, \quad \Pi = Z[k_1, L] = \begin{pmatrix} \cos k_1 L & -\sin k_1 L / k_1 \\ k_1 \sin k_1 L & \cos k_1 L \end{pmatrix}$$

and hence

$$F \equiv \Pi_{12} = \frac{-\sin k_1 L}{k_1} = 0. \quad (24)$$

With the aid of (9) this leads to the standard harmonic series

$$\omega = \frac{n\pi}{L} \sqrt{\frac{T}{\mu_1}}, \quad n = 1, 2, \dots \quad (25)$$

When the string is made up of two different density segments, we have

$$\Pi = \begin{pmatrix} \cos k_1 w_1 & -\frac{\sin k_1 w_1}{k_1} \\ k_1 \sin k_1 w_1 & \cos k_1 w_1 \end{pmatrix} \cdot \begin{pmatrix} \cos k_2 w_2 & -\frac{\sin k_2 w_2}{k_2} \\ k_2 \sin k_2 w_2 & \cos k_2 w_2 \end{pmatrix},$$

with $w_1 + w_2 = L$. Now (23) yields the condition

$$F \equiv -\cos k_1 w_1 \frac{\sin k_2 w_2}{k_2} - \frac{\sin k_1 w_1}{k_1} \cos k_2 w_2 = 0, \quad (26)$$

or equivalently with the aid of (9)

$$\sqrt{\mu_1} \tan\left(\omega w_2 \sqrt{\frac{\mu_2}{T}}\right) + \sqrt{\mu_2} \tan\left(\omega w_1 \sqrt{\frac{\mu_1}{T}}\right) = 0. \quad (27)$$

Here approximate solutions for the mode-frequencies ω may be obtained by graphical means while more accurate determinations are discussed in the next section.

4. Numerical determination of the eigenmodes

A straightforward procedure for extracting the mode-frequencies ω from the condition (23), is to select a grid of equally spaced frequencies starting at some lowest $\omega_0 > 0$ and extending it to a predetermined cut-off ω_{co} . The function $F \equiv \Pi_{12}$ is then evaluated and monitored over each of these grid-points with a mode-frequency located between any two successive grid values over which Π_{12} changes sign. In this way a series of bounds is obtained on the mode-frequencies which lie in the search range (ω_0, ω_{co}) . Each of these bounds may then be individually improved by introducing a finer grid which spans this pair of upper and lower bounds, and thus by successive refinements, any mode can be quickly located to an accuracy limited only by computer round-off errors. As a simple example, Fig. 2 shows a plot of Π_{12} versus the frequency $f (\equiv \omega/2\pi)$ over 200 initial grid points spanning the frequency range $f_0 = 0.1$ Hz to $f_{co} = 35$ Hz for the case of a two-segment string parametrized as $w_1 = w_2 = 0.5$ m, $\mu_1 = 0.01$ kg/m, $\mu_2 = 0.05$ kg/m, $T = 10$ N.

The three crossovers indicate that three eigenmodes are included in the selected frequency range and without further refinement are bounded as 1st mode: 8.65 Hz to 8.83 Hz, 2nd mode: 20.16 Hz to 20.35 Hz, 3rd mode: 29.76 Hz to 29.94 Hz. From the trend of the plot near $f = 0$ it is clear that these are indeed the

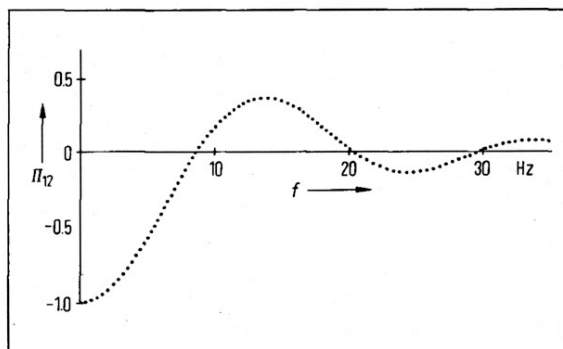


Fig. 2. A normalized plot of the function $F(\omega) \equiv \Pi_{12}$ versus frequency $f (\equiv \omega/2\pi)$ for a two segment string as parametrized in section 4. 200 equally spaced grid points in the range $f_0 = 0.1$ Hz to $f_{co} = 35$ Hz enclose the first three eigenfrequencies of the string as shown by the three zeros of F .

lowest three modes. The expected deviation from a harmonic series is obvious.

The actual machine evaluation of the function F proceeds by separately computing each 2×2 Z -matrix arising in the product (22), with one such matrix coming from each string segment. These matrices are then multiplied together and the (1,2) element of their product yields F . The individual $Z[k_j, w_j]$ have non-singular elements for all values of $k_j \equiv \omega \sqrt{\mu_j/T}$, though for certain rather extreme choices of the parameters μ_j and T , one may have $k_j \approx 0$ which then in turn requires that the (1,2)-element of Z be evaluated as a series expansion in $k_j w_j$ rather than as a quotient of two vanishing quantities. This remedy though easily encoded in a computer program, should not be needed for physically realistic strings for which $\mu_j \rightarrow 0$ and/or $T \rightarrow \infty$ are not likely possibilities.

With the mode frequency ω thus determined, it only remains to evaluate the a, b -coefficients if one also wishes to specify the string displacement functions $\psi_j(x)$ associated with that mode. This amounts to solving the series of eqs. (15) for the various a_j, b_j using the given mode frequency ω in the expression (9) for the various constants k_j . By choosing a convenient normalization in which $b_1 = 1$, we readily obtain the desired coefficients in the form

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \{Z[k_1, w_1]\}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \{Z[k_1, w_1] Z[k_2, w_2]\}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\vdots$$

$$\begin{pmatrix} a_N \\ b_N \end{pmatrix} = \{Z[k_1, w_1] Z[k_2, w_2] \dots Z[k_{N-1}, w_{N-1}]\}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (28)$$

Here each partial product of Z -matrices appearing in braces is already available without extra computation if one takes the foresight to store them during the intermediate stages of construction of the complete product Π of (22) needed for the mode frequency determination. In that case it is only necessary to invert these already computed products which, due to their unimodularity, is a trivial task. Thus writing formally $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ for any one of the products in braces, we evaluate every right hand side of (28) by the rule

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} S & -Q \\ -R & P \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -Q \\ P \end{pmatrix}. \quad (29)$$

Clearly this amounts simply to a fast look-up procedure without extra computation. The resulting overall standing wave pattern on the string is then obtained by juxtaposing the various segment functions $\psi_1(x)$, $\psi_2(x)$, ..., $\psi_N(x)$, and is exemplified in Fig. 3 where we show the wave patterns associated with the first three modes of the two-segment string as parametrized above.

5. Sample calculation for a uniformly increasing string density

As an example of a large N calculation we consider a string lying between $x = 0$ and $x = L$ with a linear density function

$$\mu(x) = \mu_L + (\mu_R - \mu_L) \frac{x}{L}. \quad (30)$$

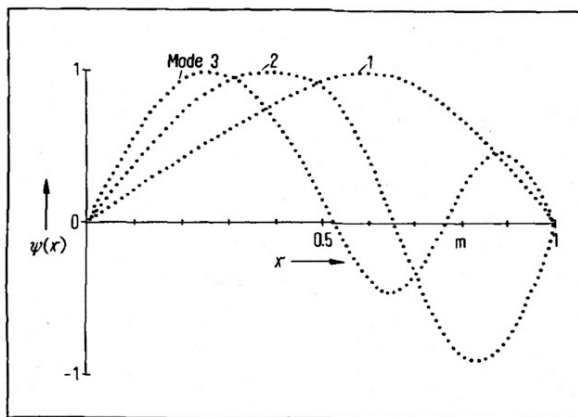


Fig. 3. Normalized plots of the standing wave patterns associated with the first three modes of the two segment string as parametrized in section 4. The left half of this 1 m string has the lower density (0.01 kg/m) with the right half 5 times as dense.

Here the selection of parameters

$$\begin{aligned} L &= 1 \text{ m}, \\ \mu_L &= 0.01 \text{ kg/m}, \\ \mu_R &= 0.04 \text{ kg/m}, \end{aligned}$$

leads to a string whose ends differ in density by a factor of 4 (equivalent to doubling the string cross-sectional diameter from one end to the other). We set the tension $T = 10 \text{ N}$ and divide the string into $N = 100$ segments of equal width $w = L/N$. Within each segment j , the density μ_j is taken as the average value

$$\mu_j \equiv \frac{1}{w} \int_{(j-1)w}^{jw} \mu(x) dx = \mu_L + \frac{(\mu_R - \mu_L)(j - \frac{1}{2})}{N}. \quad (31)$$

By selecting an initial range of frequency from $f_0 = 9 \text{ Hz}$ to a cut-off of $f_{co} = 31 \text{ Hz}$ and using 500 frequency grid points we include 3 modes. The lower and upper bounds on these mode-frequencies are listed in Table I under $r = 0$, and are followed by 3 successive refinements $r = 1, 2, 3$ each obtained by inserting 9 equally spaced grid points between the previous pair of bounds. Only meaningful significant figures are retained from the double-precision calculation with upper/lower bounds always rounded up/down. Fig. 4 shows the standing-wave patterns associated with these 3 modes. Though the wave patterns are visibly distorted from those associated with a string of uniform density, surprisingly the mode-frequencies are only slightly different from the harmonic series based on the first mode. Thus comparing with the averages of the $r = 3$ lower and upper bounds of Table I, the second and third modes are up only

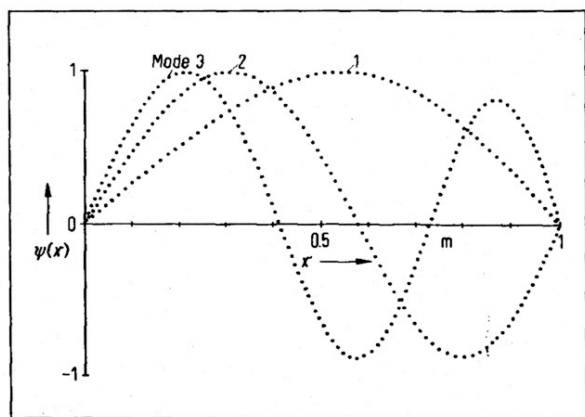


Fig. 4. Normalized standing wave patterns for the first three modes of the string described in section 5 having the uniformly increasing string density function (30).

Table I. Lower and upper bounds on the first three mode frequencies (in Hz) of the string having uniformly increasing mass density as parametrized in section 5 and approximated by $N = 100$ equal width segments. Refinement $r = 0$ designates the initial 500 grid point evaluation of F over the frequency range 9 to 31 Hz, while each succeeding refinement $r = 1, 2, 3$ inserts 9 extra grid points between the previous bounds.

Refine- ment r	Mode 1		Mode 2		Mode 3	
	Lower	Upper	Lower	Upper	Lower	Upper
	f Hz	f Hz	f Hz	f Hz	f Hz	f Hz
0	9.92	9.97	20.13	20.18	30.34	30.39
1	9.923	9.929	20.162	20.168	30.360	30.371
2	9.923	9.925	20.165	20.166	30.3703	30.3708
3	9.9242	9.9244	20.1653	20.1654	30.3707	30.3708

1.60% and 2.01% respectively on the corresponding second and third harmonics. This is quite interesting in view of the large variation in the linear mass density over the length of the string.

6. Perturbation of the mode frequencies

The frequency criterion (23) allows us to consider small perturbations $\delta\mu_1, \delta\mu_2, \dots, \delta\mu_N$ in the segment densities and to calculate the amount $\delta\omega$ by which a given mode frequency ω is shifted as a consequence. Thus assuming $\delta\omega$ and the $\delta\mu_j$ to be such related increments, we immediately calculate from (9) the shifts in wavenumber δk_j as

$$\frac{\delta k_j}{k_j} = \frac{\delta\omega}{\omega} + \frac{\delta\mu_j}{2\mu_j}, \quad j = 1, 2, \dots, N. \quad (32)$$

The required frequency shift $\delta\omega$ is then determined by the two conditions

$$\left. \begin{aligned} F(k) &= 0 \\ F(k + \delta k) &= 0 \end{aligned} \right\}, \quad (33)$$

where we now consider the function F of (23) to depend on the arguments $k \equiv (k_1, k_2, \dots, k_N)$. (The segment widths w_j are not varied in this calculation and so do not appear as arguments of F .) Working to first order in small increments and using (32), we obtain the desired linear relationship between the $\delta\omega$ and the $\delta\mu_j$ in the form

$$\frac{\delta\omega}{\omega} = \sum_{j=1}^N E_j \frac{\delta\mu_j}{\mu_j}, \quad (34)$$

with the coefficients E_j defined via

$$E_j = -\frac{D_j}{2(D_1 + D_2 + \dots + D_N)} \quad (35a)$$

with

$$D_j \equiv k_j \frac{\partial F}{\partial k_j}(k). \quad (35b)$$

To evaluate these coefficients, we first define the matrix

$$H[k, w] \equiv \quad (36)$$

$$\begin{pmatrix} -kw \sin kw & w \left(\frac{\sin kw}{kw} - \cos kw \right) \\ \frac{1}{w} (kw \sin kw + (kw)^2 \cos kw) & -kw \sin kw \end{pmatrix},$$

with the aid of which we may write, correct to first order in δk

$$Z[k + \delta k, w] = Z[k, w] + \frac{\delta k}{k} H[k, w]. \quad (37)$$

It is then readily seen on expanding the product

$$\Pi + \delta \Pi \equiv Z[k_1 + \delta k_1, w_1] \cdot Z[k_2 + \delta k_2, w_2] \dots Z[k_N + \delta k_N, w_N]$$

correct to the first order, that the D_j coefficients are identified as

$$D_j = k_j \frac{\delta \Pi_{12}}{\delta k_j} = \{Z[k_1, w_1] \dots Z[k_{j-1}, w_{j-1}] \cdot H[k_j, w_j] Z[k_{j+1}, w_{j+1}] \dots Z[k_N, w_N]\}_{12}. \quad (38)$$

Here the matrix product in braces is again easily calculated with the aid of the stored partial products of the Z -matrices (pertinent to the mode-frequency ω). Thus the partial product to the left of the H -matrix in (38) is immediately obtained and is postmultiplied by $H[k_j, w_j]$ which itself is computed according to (36). The result is then postmultiplied by the partial product to the right of H , which at this stage, though not available from store, may be easily computed using the next stored product

$$Z[k_1, w_1] \dots Z[k_j, w_j] \equiv \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \quad (39)$$

as

$$\begin{aligned} Z[k_{j+1}, w_{j+1}] \dots Z[k_N, w_N] \\ = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}^{-1} \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix} \\ = \begin{pmatrix} S & -Q \\ -R & P \end{pmatrix} \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}. \end{aligned} \quad (40)$$

The required D_j is then just the (1, 2) element of the overall product within the braces of (38).

We note that for the special case of a string of uniform density μ with small density irregularities $\delta\mu_j$, the computation of the D_j is even simpler since here all

k_j take the same value, namely $k \equiv \omega \sqrt{\mu/T}$, as a consequence of which (38) reduces to

$$D_j = \{Z[k, w_1 + w_2 + \dots + w_{j-1}] \cdot H[k, w_j] Z[k, w_{j+1} + \dots + w_N]\}_{12}$$

wherein use has been made of the additive property of the Z -matrices namely

$$Z[k, w_1] Z[k, w_2] = Z[k, w_1 + w_2].$$

If we now list the various mode frequencies as $\omega^{(i)}$ $i = 1, 2, \dots$ corresponding to the unperturbed segment densities $\mu_1, \mu_2, \dots, \mu_N$, then each $\omega^{(i)}$ will change to $\omega^{(i)} + \delta\omega^{(i)}$ as these densities are perturbed, and we shall have a series of equations of the type (34), namely

$$\frac{\delta\omega^{(i)}}{\omega^{(i)}} = \sum_{j=1}^N E_j^{(i)} \frac{\delta\mu_j}{\mu_j}, \quad i = 1, 2, \dots, \quad (41)$$

where

$$E_j^{(i)} \equiv - \frac{D_j^{(i)}}{2(D_1^{(i)} + D_2^{(i)} + \dots + D_N^{(i)})}, \quad (42a)$$

with

$$D_j^{(i)} \equiv k_j^{(i)} \frac{\partial F}{\partial k_j} (k^{(i)}) \quad (42b)$$

and

$$k_j^{(i)} \equiv \omega^{(i)} \sqrt{\frac{\mu_j}{T}}. \quad (43)$$

Thus whenever $M \leq N$ frequency shifts $\delta\omega^{(i)}$ are specified a priori, the eqs. (41) allow one to determine the corresponding density perturbations which will achieve these shifts. The $\delta\mu_j$ are uniquely specified only if $M = N$. Otherwise they have a degree of arbitrariness.

To exemplify the design implications of the above, we calculate the changes $\delta\mu_1/\mu_1, \delta\mu_2/\mu_2$ for a two-segment string which will leave the fundamental mode unchanged ($\delta\omega^{(1)} = 0$) but will shift the next mode upwards by 1% ($\delta\omega^{(2)}/\omega^{(2)} = 0.01$). Here using the two-segment string parametrization given in section 4, the two equations $i = 1, 2$ of (41) are uniquely inverted to yield

$$\frac{\delta\mu_1}{\mu_1} = -0.1418, \quad \frac{\delta\mu_2}{\mu_2} = 0.0157,$$

showing that a decrease of about 14% is required in the first segment density together with an increase of around 1.5% in the density of the second. Clearly on introducing these two density variations, the third and higher modes will also be changed in some unspecified way, but we cannot have control over these in addition, unless more than $N = 2$ segments are introduced.

7. Discussion

The above analysis based on the method of segments represents a quick and easy way to study the vibration modes of strings of non-uniform mass density. With the analytic aspects of the boundary value problem exactly factored out, one is left with a residual problem of pure algebra which when encoded in a computer program, yields the various eigenmodes directly via the zeros of the known transcendental function $F(\omega)$. No expertise in the numerical solution of differential equations is required by this approach, which when fully implemented allows one to experiment freely with a whole variety of string density profiles to gain a feel of their corresponding mode structures.

Of particular interest for musical instrument design is the possibility of generating second order beats on one string only. Normally these come from two separate sources slightly off-tuned from the octave (which is the most important perfect interval for this type of beating) but are readily produced on a single string of non-uniform density whenever the second mode frequency is a few Hertz different from twice the first. A simple way to achieve this is by appropriately loading a conventional string of uniform density. As an example our calculation shows that when a 1-metre string of uniform density 0.01 kg/m is additionally loaded to 0.10 kg/m over the region $x = 20$ cm to $x = 21$ cm measured from its end, then with a tension of 50 Newton, the second mode frequency ($= 65.11$ Hz) is 3.1 Hz below the second harmonic of the first ($= 34.16$ Hz). Based on this idea our preliminary experiments with loaded sonometer strings show that such "built-in" second order beating can be quite audible and not at all unpleasant.

Reference

- [1] Kalotas, T. M., Lee, A. R., A new approach to one-dimensional scattering. *Amer. J. Phys.* **59** [1991], 48.